A Novel Error-Reducing Approach on the Fast Fourier Transform Option Valuation

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Abstract

Applying the fast Fourier transform (FFT) for pricing derivatives is one of the popular and important evaluation methodologies. A general, and highly cited FFT-based approach proposed by Carr and Madan (1999) can efficiently price vanilla options given that the characteristic function of the underlying asset's return is analytically known. However, their pricing results converge slowly and even are negative for deep-out-of-the-money options. This thesis proposes a novel approach to address these problems. My approach decomposes the option value into the proxy and residual terms: The proxy term approximates the theoretical option value and can be analytically evaluated without generating numerical error; that is why my approach can generate less pricing error. The residual term numerically estimates the difference between the theoretical option value and the proxy term. Numerical experiments suggest that my superior approach efficiently reduces the pricing error and alleviates the negative price problem for evaluating deep-out-of-the-money options.

JEL classification: C02; G13

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1 Introduction

Various stochastic processes, for example, the jump-diffusion model studied by Merton (1976) and the CGMY model studied by Carr et al. (2002), are proposed to properly fit the empirical phenomena, says, like the high pick and heavy tail, of the underlying asset's price process. These complicated stochastic processes would make the derivative pricing problems become intractable. Fortunately, the characteristic functions for a vast class of stochastic processes used for modeling asset returns are simple and the securities can be numerically evaluated by taking advantages of the Fourier analysis (see e.g. Heston (1993), Bakshi and Madan (2000)). Carr and Madan (1999) introduce the fast Fourier transform (FFT hereafter) into this class of pricing approaches and their work is widely adopted during the past few years (see e.g. Carr and Wu (2003), Lee (2004)).

Carr and Madan (1999) show that the option value can be expressed as the inverse Fourier transform of the Fourier transform of the damped option price, which can be expressed in terms of the characteristic function of the logarithm of the underlying asset's price and is the product of the option price and an exponential function. The latter characteristic ensures that the Fourier transform of the damped option price is well-defined, and can be numerically determined by the FFT. However, the integrand of this quadrature is usually highly oscillatory and consequently great computational cost is needed in order to get accurate pricing results. This paper improves the option valuation method studied by Carr and Madan (1999), which is widely cited and applied in academic literature, by significantly reducing the pricing error.

Assume that the price of an underlying asset of vanilla options follows a target stochastic process G for convenience. To reduce the pricing error, the key idea of the proposed approach is to split the Fourier transform of the damped option price (used in Carr and Madan (1999)) into the proxy and residual parts. The proxy part is chosen as the the Fourier transform of the damped option price implied by another stochastic process G' (called the"proxy process" hereafter) that satisfies the following constraints: (i) analytical formulas for both the characteristic function of G' and the vanilla option values under G' are admitted. (ii) The Fourier transform of the damped option price under G is close to that under G'. Besides, the residual part is defined as the difference of the two Fourier transforms mentioned above. The second constraint can be achieved by calibrating the cumulants of underlying asset's price of process G' to match the cumulants of underlying asset's price of G. With this decomposition, the theoretical option value under G can be split into the proxy term, the option value under G', and the residual term, which is the difference of the theoretical option value and the proxy term price. The proxy term can be obtained with an analytic formula and consequently contributes to no numerical pricing error. Only the residual term has to be numerically evaluated with the FFT and introduces numerical error. Therefore, the total pricing error is significantly reduced. In this thesis, the proxy term of the option value is chosen as the option price implied by the jump-diffusion model studied by Merton (1976) since it can be expressed as a quickly converging series of the option price implied by the diffusion model studied by Black and Scholes (1973). Note that any other process that satisfies the first constraint can be used as the proxy process without damaging our framework. Our numerical experiments take the option pricing under the variance gamma model studied by Madan et al. (1998), the stochastic volatility model studied by Heston (1993), and the double exponential model studied by Kou (2002) as examples to demonstrate the superiority of our approach.

Furthermore, the original FFT option valuation suggested by Carr and Madan (1999) may generate inaccurate or even negative pricing results for deep-out-of-the-money options unless they use extremely finer numerical integration(see e.g. Carr and Madan (2009)). Our methodology also alleviates this problem.

The remainder of this paper is organized as follows. Section 2 reviews the FFT option valuation approach studied by Carr and Madan (1999) and the jump-diffusion model studied by Merton (1976). Section 3 introduces how our approach could reduce the pricing error by decomposing the option value into the proxy and residual terms. The way to calibrate the parameters of the proxy stochastic process G' to suppress the differences between the densities of the underlying asset's return under G and G' is also discussed in this section. Numerical results in section 4 verify the superiority of our approach. Section 5 concludes.

2 Preliminaries

2.1 Characteristic Function and Cumulants

Let X be a real-valued random variable, the characteristic function for X is defined as

$$\phi_X(u) = \mathbb{E}\left[e^{iux}\right] = \int_{-\infty}^{\infty} e^{iux}\, dF_X(x),$$

where F_X is the cumulative distribution function of X.

Moreover, if ϕ_X is k times differentiable, then the k^{th} cumulant for X is defined as

$$(-i)^k (\log \phi_X)^{(k)}(0).$$

Note that $\log \phi_X$ is also named as the cumulant-generating function.

2.2 Fast Fourier Transform (FFT)

The FFT is an efficient algorithm to calculate discrete Fourier transform. It improves the time complexity for calculating the following sum

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}\chi(j) \quad \text{for} \quad k = 1, 2, \ldots, N, \tag{1}$$

from $O(N^2)$ to $O(N \log_2 N)$, where N is usually a power of 2; χ is a real-valued integrable function.

2.3 Composite Simpson's Rule

The composite Simpson's rule is a method of numerical integration. Let f be a real-valued integrable function on interval $[a, b]$ with continuous forth derivative and n be an even number, there is a $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, dx = \frac{h}{3}[f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(x_n)] - \frac{b-a}{180}h^4 f^{(4)}(\xi), \tag{2}$$

where n denotes an even number, $x_j = a + jh$ for $j = 0, 1, ..., n$, and $h = (b-a)/n$. Eq. (2) suggests that the definite integrals can be approximated by

$$\int_a^b f(x)\, dx = \frac{h}{3}[f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(x_n)].$$

with quadrature error bounded above by

$$\frac{b-a}{180} h^4 \sup_{y \in [a,b]} \left| f^{(4)}(y) \right|. \tag{3}$$

2.4 Carr and Madan's Option Pricing Method

Denote by $C_T(k)$ the price of a vanilla call option with maturity date T and strike price $K := e^k$. Let $\phi_T(v)$ and $q_T(s)$ denote the characteristic function and the risk-neutral density of the logarithm of the underlying asset price at option maturity. Carr and Madan (1999) suggest that

$$C_T(k) := \int_k^\infty (e^s - e^k) q_T(s)\, ds = \frac{e^{-dk}}{\pi} \int_0^\infty e^{-ivk} \psi_T(v)\, dv, \tag{4}$$

where $\psi_T(v)$ is the Fourier transform of the damped call price $c_T(k) \equiv e^{dk} C_T(k)$ for some damping constant $d > 0$, which can be further expressed in terms of $\phi_T(v)$ as follows:

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^\infty e^{ivk} c_T(k)\, dk = \int_{-\infty}^\infty e^{ivk} e^{dk} C_T(k)\, dk \\
&= \int_{-\infty}^\infty e^{ivk} e^{dk} e^{-rT} \int_k^\infty (e^s - e^k) q_T(s)\, ds\, dk \\
&= e^{-rT} \int_{-\infty}^\infty q_T(s) \left[\frac{e^{k(iv+d)+s}}{iv+d} - \frac{e^{k(iv+d+1)}}{iv+d+1} \right]_{k=-\infty}^{k=s} ds \\
&= \frac{e^{-rT}\, \phi_T(v - (d+1)i)}{d^2 + d - v^2 + i(2d+1)v},
\end{aligned} \tag{5}$$

where r denotes the risk-free interest rate. Since the Fourier transform of $C_T(k)$ does not exist due to the fact that $C_T(k)$ converges to the current value of the underlying asset as $k \to -\infty$, we have to choose a proper damping constant $d > 0$ to make sure that $e^{dk} C_T(k)$ is integrable, and therefore the Fourier transform of $c_T(k)$ is well-defined. In this work, we use $d = 1.5$ as the damping constant, as suggested by Carr and Madan (1999).

If $\phi_T(v)$ (and hence $\psi_T(v)$) is known analytically, the call value given in Eq. (4) can be approximated by the following sum

$$C_T(k) \approx \frac{e^{-dk}}{\pi} \sum_{j=1}^N e^{-iv_j k} \psi_T(v_j) \eta, \tag{6}$$

where $v_j \equiv \eta(j-1)$; η is the distance between quadrature points; N is the number of quadrature points. Therefore, the effective upper limit for the integration, which is defined as the truncation point of the integrand, is equal to $N\eta$. In this work, we choose $N\eta$ as 1024, as in Carr and Madan (1999). This integration can be efficiently carried out by the FFT introduced in Sec. 2.1.

Carr and Madan (1999) use the FFT to simultaneously evaluate the values of N otherwise identical options with different strike prices. The logarithms of these strike prices can be defined as

$$k_u \equiv k_1 + \Lambda(u-1) \quad \text{for} \quad u = 1, 2, 3, \ldots, m, \tag{7}$$

where k_1 and Λ are properly selected constants to make the interval $[k_1, k_m]$ contain all strike prices of the option contracts in which we are interested. Substituting Eq. (7) into Eq. (6) yields

$$\begin{aligned} C_T(k_u) &\approx \frac{e^{-dk_u}}{\pi} \sum_{j=1}^{N} e^{-iv_j(k_1+\Lambda(u-1))} \psi_T(v_j)\eta \\ &= \frac{e^{-dk_u}}{\pi} \sum_{j=1}^{N} e^{-i\Lambda\eta(j-1)(u-1)} e^{-ik_1 v_j} \psi_T(v_j)\eta. \end{aligned} \tag{8}$$

Note that both Λ and η should be properly selected to satisfy the constraint

$$\Lambda\eta = \frac{2\pi}{N}. \tag{9}$$

Thus, Eq. (8) can be expressed in terms of Eq. (1), and therefore can be efficiently evaluated by the FFT.

2.5 Merton's Jump-Diffusion Model

Our paper uses the jump-diffusion model studied by Merton (1976) to play the role of the proxy process because the characteristic functions of both the logarithm of the underlying asset's price and the damped option value under the process are analytically known. Note that other suitable model can serve as the proxy process without difficulty if analytical formulas for both the characteristic function of the logarithm of the underlying asset's price and the vanilla option values under it are admitted. Now we give a brief review of Merton's jump-diffusion model.

Under Merton's jump-diffusion model, the asset price dynamics under some measure $\mathbb{P}$ are

$$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t\, dt + \sigma S_t\, dW_t + (e^{J_t} - 1) S_t\, dM_t, \tag{10}$$

where μ and σ^2 are respectively the instantaneous mean and variance of the return conditional on the Poisson event does not occur; W_t is a standard Brownian motion; M_t is a Poisson process with intensity λ; J_t is an independent normally distributed random variable with mean α and standard derivation β. By applying the Itô's formula, the differential form of the log price process can be expressed as

$$\begin{aligned} d\ln S_t &= \frac{1}{S_t}\left[\left(\mu + \frac{\sigma^2}{2}\right) S_t\, dt + \sigma S_t\, dW_t\right] - \frac{1}{2S_t^2}\sigma^2 S_t^2\, dt + \ln\left[(e^{J_t} - 1)S_t + S_t\right] - \ln S_t \\ &= \mu\, dt + \sigma\, dW_t + J_t\, dM_t. \end{aligned} \tag{11}$$

Denote by f the density of the logarithm of the underlying asset implied by Eq. (11). According to the law of total probability, f can be expressed as

$$\begin{aligned} f(\ln S_T) &= \sum_{j=0}^{\infty} \mathbb{P}(M_T = j) f(\ln S_T | M_T = j) \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \frac{1}{\sqrt{2\pi(\sigma^2 T + j\beta^2)}} \exp(-\frac{(\ln S_T - (\ln S_0 + \mu T + j\alpha))^2}{2(\sigma^2 T + j\beta^2)} \end{aligned} \tag{12}$$

Moreover, Eq. (11) implies that

$$\ln(\frac{S_T}{S_0}) = \mu T + \sigma W_T + \sum_{k=0}^{M_T} J_k. \tag{13}$$

Therefore, the terminal underlying asset's price can be represented as

$$S_T = S_0 \exp\left(\mu T + \sigma W_T + \sum_{k=0}^{M_T} J_k\right). \tag{14}$$

Hence, the expected growth of the underlying asset under measure $\mathbb{P}$ will be

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[S_T] &= S_0\, \mathbb{E}_{\mathbb{P}}[e^{\mu T}]\, \mathbb{E}_{\mathbb{P}}[e^{\sigma W_T}]\, \mathbb{E}_{\mathbb{P}}\left[e^{\sum_{k=0}^{M_T} J_k}\right] \\ &= S_0\, \mathbb{E}_{\mathbb{P}}[e^{\mu T}]\, \mathbb{E}_{\mathbb{P}}[e^{\sigma W_T}]\, \mathbb{E}_{M_T}\mathbb{E}_{\mathbb{P}}\left[\prod_{k=0}^{n} e^{J_k} | M_T = n\right] \\ &= S_0\, \mathbb{E}_{\mathbb{P}}[e^{\mu T}]\, \mathbb{E}_{\mathbb{P}}[e^{\sigma W_T}]\, \mathbb{E}_{M_T}\left[e^{(\alpha + \frac{\beta^2}{2})M_T}\right] \\ &= S_0\, e^{\mu T}\, e^{\frac{\sigma^2}{2}T}\, e^{\lambda T(e^{\alpha + \frac{\beta^2}{2}} - 1)} \\ &= S_0\, \exp\left(\left(\mu + \frac{\sigma^2}{2} + \lambda(e^{\alpha + \frac{\beta^2}{2}} - 1)\right) T\right). \end{aligned} \tag{15}$$

Eq. (15) implies that the discount rate under measure $\mathbb{P}$ must be

$$\mu + \frac{\sigma^2}{2} + \lambda(e^{\alpha+\frac{\beta^2}{2}} - 1).$$

Besides, the characteristic function for the logarithm of S_T implied by the above stochastic differential equation in Eq. (11) is

$$\begin{aligned}
&\phi_T^{\mathrm{MJD}}(u) := \mathbb{E}_{\mathbb{P}}(e^{iu\ln S_T}) = \int_{-\infty}^{\infty} e^{iu\ln S_T} f(\ln S_T)\, d\ln S_T \\
= \;& \int_{-\infty}^{\infty} e^{iu\ln S_T} \lim_{n\to\infty} \sum_{j=0}^{n} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \frac{\exp\left(-\frac{(\ln S_T - (\ln S_0 + \mu T + j\alpha))^2}{2(\sigma^2 T + j\beta^2)}\right)}{\sqrt{2\pi(\sigma^2 T + j\beta^2)}}\, d\ln S_T \qquad (16) \\
= \;& \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{iu\ln S_T} \sum_{j=0}^{n} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \frac{\exp\left(-\frac{(\ln S_T - (\ln S_0 + \mu T + j\alpha))^2}{2(\sigma^2 T + j\beta^2)}\right)}{\sqrt{2\pi(\sigma^2 T + j\beta^2)}}\, d\ln S_T \\
= \;& \lim_{n\to\infty} \sum_{j=0}^{n} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \int_{-\infty}^{\infty} e^{iu\ln S_T} \frac{\exp\left(-\frac{(\ln S_T - (\ln S_0 + \mu T + j\alpha))^2}{2(\sigma^2 T + j\beta^2)}\right)}{\sqrt{2\pi(\sigma^2 T + j\beta^2)}}\, d\ln S_T \\
= \;& \lim_{n\to\infty} \sum_{j=0}^{n} \frac{e^{-\lambda T}(\lambda T)^j}{j!} \exp\left(iu(\ln S_0 + \mu T + j\alpha) - \frac{(\sigma^2 T + j\beta^2)u^2}{2}\right) \qquad (17) \\
= \;& \exp\left(-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2\right) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(\lambda T \exp(iu\alpha - \frac{\beta^2 u^2}{2}))^j}{j!} \qquad (18) \\
= \;& \exp\left(iu\ln S_0 + \left(iu\mu - \frac{\sigma^2 u^2}{2} + \lambda\left(e^{iu\alpha - \frac{\beta^2 u^2}{2}} - 1\right)\right) T\right).
\end{aligned}$$

Eq. (17) holds since according to the monotone convergence theorem, we can interchange the integral operator with the limit operator in Eq. (16). Besides, the first five cumulants for the logarithm of S_T implied by Merton's jump-diffusion model are

$$\begin{aligned}
1^{st}\text{cumulant} \;&:=\; (-i)(\log \phi_T^{\mathrm{MJD}})'(0) = \ln S_0 + (\mu + \lambda\alpha)\, T; \\
2^{nd}\text{cumulant} \;&:=\; (-i)^2(\log \phi_T^{\mathrm{MJD}})''(0) = \left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right) T; \\
3^{rd}\text{cumulant} \;&:=\; (-i)^3(\log \phi_T^{\mathrm{MJD}})'''(0) = \lambda(\alpha^3 + 3\alpha\beta^2)T; \\
4^{th}\text{cumulant} \;&:=\; (-i)^4(\log \phi_T^{\mathrm{MJD}})^{(4)}(0) = \lambda(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4)T; \\
5^{th}\text{cumulant} \;&:=\; (-i)^5(\log \phi_T^{\mathrm{MJD}})^{(5)}(0) = \lambda(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4)T.
\end{aligned}$$

Moreover, the vanilla call price implied by Merton's jump-diffusion model with strike price $K := e^k$ and maturity T is equal to

$$
\begin{aligned}
C_T^{\mathrm{MJD}}(k) &= e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\int_k^{\infty}(e^{\ln S_T}-e^k)f(\ln S_T)\,d\ln S_T \\
&= e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\int_k^{\infty}(e^{\ln S_T}-e^k)\lim_{n\to\infty}\sum_{j=0}^{n}\frac{e^{-\lambda T}(\lambda T)^j}{j!}\frac{\exp(-\frac{(\ln S_T-(\ln S_0+\mu T+j\alpha))^2}{2(\sigma^2T+j\beta^2)})}{\sqrt{2\pi(\sigma^2T+j\beta^2)}}d\ln S_T \\
&= e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\lim_{n\to\infty}\sum_{j=0}^{n}\frac{e^{-\lambda T}(\lambda T)^j}{j!}\int_k^{\infty}(e^{\ln S_T}-e^k)\frac{\exp(-\frac{(\ln S_T-(\ln S_0+\mu T+j\alpha))^2}{2(\sigma^2T+j\beta^2)})}{\sqrt{2\pi(\sigma^2T+j\beta^2)}}d\ln S_T \quad (19)
\end{aligned}
$$

Again, Eq. (19) holds because we use the monotone convergence theorem to interchange the limit and integral operators. Now we set

$$
Q := \frac{\ln S_T-(\ln S_0+\mu T+j\alpha)}{\sqrt{\sigma^2T+j\beta^2}}
$$

and

$$
k' := \frac{k-(\ln S_0+\mu T+j\alpha)}{\sqrt{\sigma^2T+j\beta^2}}.
$$

According to the change of variable theorem, Eq. (19) can be rewritten as

$$
\begin{aligned}
C_T^{\mathrm{MJD}}(k) &= e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\sum_{j=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^j}{j!}\int_{k'}^{\infty}(S_0\,e^{\mu T+j\alpha+Q\sqrt{\sigma^2T+j\beta^2}}-e^k)\frac{e^{-\frac{Q^2}{2}}}{\sqrt{2\pi}}\,dQ \\
&= \sum_{j=0}^{\infty}\frac{e^{-\lambda e^{\alpha+\frac{\beta^2}{2}}T}\left(\lambda e^{\alpha+\frac{\beta^2}{2}}T\right)^j}{j!}\left(S_0\int_{k'}^{\infty}\frac{e^{-\frac{(Q-\sqrt{\sigma^2T+j\beta^2})^2}{2}}}{\sqrt{2\pi}}\,dQ-e^ke^{-(\mu+\frac{\sigma^2}{2})T-j(\alpha+\frac{\beta^2}{2})}\int_{k'}^{\infty}\frac{e^{-\frac{Q^2}{2}}}{\sqrt{2\pi}}\,dQ\right) \\
&= \sum_{j=0}^{\infty}\frac{e^{-\lambda' T}(\lambda' T)^j}{j!}C_T^{\mathrm{BS}}(S_0,k,\sigma_j,r_j), \qquad (20)
\end{aligned}
$$

where $\lambda' := \lambda e^{\alpha+\frac{\beta^2}{2}}$; $C_T^{\mathrm{BS}}(S_0,k,\sigma_j,r_j)$ denotes the Black-Scholes call option pricing formula with underlying asset's price S_0, strike price $K := e^k$, time to maturity T, risk-free rate

$$
r_j := \mu+\frac{\sigma^2}{2}+\frac{j(\alpha+\frac{\beta^2}{2})}{T},
$$

and volatility rate

$$
\sigma_j := \sqrt{\sigma^2+\frac{j\beta^2}{T}}.
$$

2.6 Variance Gamma Model

The variance gamma model (VG model hereafter) studied by Madan et al. (1998) is a generalized Brownian motion. It can be obtained by evaluating Brownian motion with drift at a random time change given by gamma process.

A gamma process is a random process with independent gamma distributed increments. Let $\gamma_t(\delta, \nu)$ be a gamma process with mean rate δ and variance rate ν. The increment $\gamma_{t+h}(\delta, \nu) - \gamma_t(\delta, \nu)$ follows gamma distribution with mean δh and variance νh.

Let W_t be a standard Brownian motion and $b_t(\vartheta, \varsigma) := \vartheta t + \varsigma W_t$ be a Brownian motion with drift rate ϑ and variance rate ς. The VG process can be defined as

$$X_t(\varsigma, \nu, \vartheta) := b_{\gamma_t(1,\nu)}(\vartheta, \varsigma),$$

where $\gamma_t(1, \nu)$ is a gamma process with mean rate 1 and variance rate ν. Moreover, the characteristic function of the the VG process is

$$\phi_{X_T}(u) := \mathbb{E}[e^{iuX_T(\varsigma,\nu,\vartheta)}] = \left(1 - i\vartheta\nu u + \frac{\varsigma^2\nu u^2}{2}\right)^{-\frac{T}{\nu}}.$$

Let r be the risk-free interest rate. If we model the asset dynamics by

$$S_T = S_0 \exp(rT + X_T(\varsigma, \nu, \vartheta) + \omega T) \tag{21}$$

and set ω as $-\ln \phi_{X_T}(-i)/T$, we get the asset price process under the risk-neutral measure. Furthermore, the characteristic function of the logarithm of the asset price implied by the risk-neutralized dynamics in Eq. (21) is

$$\begin{aligned} \phi_T^{\mathrm{VG}}(u) &:= \mathbb{E}[e^{iu\ln S_T}] = \exp(iu \ln S_0) \exp(iu(rT - \ln \phi_{X_T}(-i)))\phi_{X_T}(u) \\ &= \exp(iu(\ln S_0 + (r + (1/\nu)\ln(1 - \vartheta\nu - \varsigma^2\nu/2)T)) \left(1 - i\vartheta\nu u + \frac{\varsigma^2\nu u^2}{2}\right)^{-\frac{T}{\nu}}. \end{aligned}$$

2.7 Heston's Stochastic Volatility Model

The model studied by Heston (1993) generalizes the Black-Scholes model to a stochastic volatility version. Heston (1993) assumes that the asset dynamics follows

$$dS_t = \zeta S_t \, dt + \sqrt{v_t} S_t \, dW_t,$$

where ζ and $v_t$is the instantaneous mean and variance of the return; W_t is a standard Brownian motion. The instantaneous volatility v_t follows

$$dv_t = \kappa(\theta - v_t) \, dt + \epsilon\sqrt{v_t} \, dB_t, \tag{22}$$

where B_t is a standard Brownian motion which has correlation ρ with W_t; κ determines the mean reversion speed; θ is the long-run variance and ϵ is the volatility of volatility parameter. Note that Eq. (22) is the famous C-I-R square-root process studied by Cox et al. (1985).

The risk-neutralized characteristic function of the logarithm of the asset price implied by the Heston model is

$$\phi_T^{\text{Heston}}(u) := \mathbb{E}[e^{iu\ln S_T}] = \exp(iu(\ln S_0 + rT) + C(u) + D(u)V_0),$$

where V_0 is the current variance;

$$\begin{aligned}
d(u) &:= \sqrt{(\rho\epsilon ui - \kappa)^2 + \epsilon^2(ui + u^2)};\\
g(u) &:= \frac{\kappa - \rho\epsilon ui + d(u)}{\kappa - \rho\epsilon ui - d(u)};\\
C(u) &:= \frac{\kappa\theta}{\epsilon^2}\left[(\kappa - \rho\epsilon ui + d(u))T - 2\log\left(\frac{1 - g(u)e^{d(u)T}}{1 - g(u)}\right)\right];\\
D(u) &:= \frac{\kappa - \rho\epsilon ui + d(u)}{\epsilon^2}\left(\frac{1 - e^{d(u)T}}{1 - g(u)e^{d(u)T}}\right).
\end{aligned}$$

2.8 Kou's Jump-Diffusion Model

Kou (2002) assumes that the underlying asset price dynamics follows

$$dS_t = \left(\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}\right)S_t\,dt + \tilde{\sigma}S_t\,dW_t + (e^{Z_t} - 1)S_t\,dM_t,$$

where $\tilde{\mu}$ and $\tilde{\sigma}^2$ are respectively the instantaneous mean and variance of the return conditional on the Poisson event does not occur; W_t is a standard Brownian motion; M_t is a Poisson process with intensity $\tilde{\lambda}$; Z_t is an independent double exponential (DE hereafter) random variable with the density

$$f_Z(z) = p \cdot \omega_1 e^{-\omega_1 z} 1_{\{z \geq 0\}} + q \cdot \omega_2 e^{-\omega_2 z} 1_{\{z \leq 0\}},$$

where $\omega_1 > 0$ and $\omega_2 > 0$ represents the rate of two independent exponential random variable respectively; $p, q \geq 0$ represent the probabilities of upward and downward jumps respectively, subject to $p + q = 1$.

The risk-neutralized characteristic function of the logarithm of the asset price implied by Kou's model is

$$\begin{aligned}
&\phi_T^{\text{DE}}(u) := \mathbb{E}[e^{iu\ln S_T}]\\
&= \exp\left(iu\ln S_0 + \left(iu\left(r - \tilde{\lambda}\left(\frac{q\,\omega_2}{\omega_2 + 1} + \frac{p\,\omega_1}{\omega_1 - 1} - 1\right) - \frac{\tilde{\sigma^2}}{2}\right) - \frac{\tilde{\sigma}^2 u^2}{2} + \tilde{\lambda}\left(\frac{q\,\omega_2}{\omega_2 + iu} + \frac{p\,\omega_1}{\omega_1 - iu} - 1\right)\right)T\right).
\end{aligned}$$

3 Error Reduction with a Proxy

3.1 Decomposition of Call Price

With a suitable function $\psi_T^{\text{proxy}}(v)$ which closely approximates $\psi_T(v)$, the theoretical call value in Eq. (4) can be further decomposed into a proxy part and a residual part as follows

$$
\begin{aligned}
C_T(k) &= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T(v)\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}[\psi_T^{\text{proxy}}(v)+\psi_T^{\text{residual}}(v)]\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{proxy}}(v)\,dv + \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{residual}}(v)\,dv \\
&= C_T^{\text{proxy}}(k)+C_T^{\text{residual}}(k).
\end{aligned}
\tag{23}
$$

There are two critical criteria for choosing a sound $\psi_T^{\text{proxy}}(v)$. First, analytical formulas for both $\psi_T^{\text{proxy}}(v)$ and $C_T^{\text{proxy}}(k)$ are admitted. Without an analytical formula of $\psi_T^{\text{proxy}}(v)$, it will be very difficult to numerically determine the Fourier inversion of $\psi_T^{\text{residual}}(v)$ because we don't even have an analytic formula of $\psi_T^{\text{residual}}(v)$. If $C_T^{\text{proxy}}(k)$ can be calculated with an explicit formula, we can obtain it without performing a numerical integration. Accordingly, the proxy term contributes to zero numerical pricing error; only the residual term, which is the difference of the theoretical option value and the proxy term price, has to be numerically evaluated with the FFT in the inversion stage. Second, we choose a proxy term of the Fourier transform of the damped option value, $\psi_T^{\text{proxy}}(v)$, that is closed to $\psi_T(v)$ so that the value of $\psi_T^{\text{residual}}(v)$ is quite small. As a result, the value of the 4th order derivative of $\psi_T^{\text{residual}}(v)$ will have a better chance to be small. According to Eq. (3), the quadrature error is bounded above by the supremum of the 4th derivative of the integrand. Therefore, our method will have a better chance to generate less quadrature error than the original Carr and Madan's method since the upper bound of quadrature error has been significantly improved.

3.2 Use Merton's Jump-Diffusion Model as a Proxy

We will use Merton's jump-diffusion model to play the role of proxy. According to Eq. (18), the characteristic function of the logarithm of the call price can be expressed as an infinite series

$$\phi_T^{\text{MJD}}(u) = \exp\left(-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2\right)\sum_{j=0}^{\infty}\frac{(\lambda T \exp(iu\alpha - \frac{\beta^2 u^2}{2}))^j}{j!}.$$

Let H be a positive integer. Now the characteristic function above can be split into two parts:

$$\phi_T^{\text{MJD, 1}}(u) := \exp\left(-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2\right)\sum_{j=0}^{H-1}\frac{(\lambda T \exp(iu\alpha - \frac{\beta^2 u^2}{2}))^j}{j!},$$
$$\phi_T^{\text{MJD, 2}}(u) := \exp\left(-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2\right)\sum_{j=H}^{\infty}\frac{(\lambda T \exp(iu\alpha - \frac{\beta^2 u^2}{2}))^j}{j!}.$$

Hence, $\psi_T^{\text{MJD}}(v)$ can also be split into two parts:

$$\psi_T^{\text{MJD, 1}}(v) := \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{\text{MJD, 1}}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v},$$
$$\psi_T^{\text{MJD, 2}}(v) := \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{\text{MJD, 2}}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v}.$$

Therefore, the target call price can be divided into

$$\begin{aligned}
C_T(k) &= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T(v)\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}[\psi_T^{\text{MJD}}(v)+\psi_T^{\text{RESIDUAL}}(v)]\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\left[\psi_T^{\text{MJD,1}}(v)+\psi_T^{\text{MJD, 2}}(v)\right]\,dv + \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{RESIDUAL}}(v)\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{MJD, 1}}(v)\,dv + \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\left[\psi_T^{\text{MJD, 2}}(v)+\psi_T^{\text{RESIDUAL}}(v)\right]\,dv \\
&:= C_T^{\text{proxy}}(k) + C_T^{\text{residual}}(k),
\end{aligned}$$

where $\psi_T^{\text{RESIDUAL}}(v)$ is defined as the difference of $\psi_T(v)$ and $\psi_T^{\text{MJD}}(v)$. We use the first H terms of Merton's characteristic function to be the proxy instead of using the complete $\psi_T^{\text{MJD}}(v)$ since we want to avoid invoking infinitely many Black-Scholes formulas.

Therefore, the proxy characteristic function equals

$$\phi_T^{\text{proxy}}(u) = \exp\left(-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2\right)\sum_{j=0}^{H-1}\frac{(\lambda T \exp(iu\alpha - \frac{\beta^2 u^2}{2}))^j}{j!},$$

and therefore the proxy term of the Fourier transform of the damped option value will be

$$\psi_T^{\text{proxy}}(v) = \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{\text{proxy}}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v}.$$

Moreover, the proxy term of the theoretical call price is equal to

$$C_T^{\text{proxy}}(k) = \sum_{j=0}^{H-1} \frac{e^{-\lambda' T}(\lambda' T)^j}{j!} C_T^{\text{BS}}(S_0, k, \sigma_j, r_j).$$

Now we want to demonstrate how to calibrate corresponding parameters for the proxy characteristic function. Suppose that there is a target distribution with analytic characteristic function of the logarithm of the underlying asset's price. Since the characteristic functions (for a vast class) admit infinite differentiability (see e.g. Bakshi and Madan (2000)), we can use the method described in Sec. 2.1 to obtain the first five cumulants of the logarithm of the underlying asset's price implied by the target model. Note that we use the first five cumulants to find the corresponding parameters for Merton's model since there are only five parameters, say, five degrees of freedom, in Merton's model. Assume the first five cumulants of the logarithm of the underlying asset's price implied by the target process are known and equal to $m_1 + \ln S_0$, m_2, m_3, m_4, and m_5, respectively. The key idea is that we match the first five cumulants for the logarithm of the underlying asset's price implied by the target distribution with the first five cumulants for the logarithm of the underlying asset's price implied by Merton's jump-diffusion model to suppress the differences between the proxy and target characteristic functions. This yields

$$(\mu + \lambda\alpha)\, T = m_1, \tag{24}$$

$$\left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right) T = m_2, \tag{25}$$

$$\lambda(\alpha^3 + 3\alpha\beta^2)T = m_3, \tag{26}$$

$$\lambda(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4)T = m_4, \tag{27}$$

$$\lambda(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4)T = m_5. \tag{28}$$

Since the time to maturity is a known constant and the first five moments are multiples of T, we set $T = 1$ W.L.O.G. for notation simplicity. Note that if $T \neq 1$, we can exchange m_i with m_i/T for $i = 1, 2, \cdots, 5$ in the following derivation.

We first rewrite Eq. (26) as

$$\beta^2 = \frac{m_3 - \lambda\alpha^3}{3\lambda\alpha} \tag{29}$$

and then replace all the β^2 in Eq. (27) and Eq. (28) by Eq. (29). This yields

$$-2\alpha^6\lambda^2 + (4m_3\alpha^3 - 3\alpha^2 m_4)\lambda + m_3^2 = 0,$$
$$-2\alpha^6\lambda^2 - 3\alpha m_5\lambda + 5m_3^2 = 0.$$

By equating the above two equations, we get

$$\lambda = \frac{4m_3^2}{3m_5\alpha - 3m_4\alpha^2 + 4m_3\alpha^3}. \tag{30}$$

Now we can replace the λ and β^2 in Eq. (26) by Eq. (29) and Eq. (30), respectively, and get a polynomial equation of α as follows

$$48\alpha^4 m_3^4 - 120\alpha^3 m_3^3 m_4 + 9\alpha^2 m_3^2\left(8m_3m_5 + 5m_4^2\right) - 54\alpha m_3^2 m_4 m_5 + 9m_3^2 m_5^2 = 0. \tag{31}$$

Clearly, the variable α in Eq. (31) can be easily solved since the left hand side of Eq. (31) is a polynomial function of α with degree 4. If there are two or more roots among the real numbers, we pick the one which implies smaller λ because we want the proxy option value plays a major role in the overall call price. Note that we invoke the first H terms of the call price implied by Merton's model as the the proxy part call price. Therefore, we choose the answer with smaller λ since in this case, the probability mass of the Poisson distribution will be more concentrated on the first H terms. However, it is possible that Eq. (31) does not have any roots among the real numbers. In this case, we simply pick a proper α to minimizes the absolute value of the left hand side of Eq. (31). Once α is determined, the other 4 parameters, λ, μ, β, and σ, can also be determined easily.

Figure 3.2 provides an inspirational example for our method by the VG model. The solid line represents the probability density of the logarithm of terminal underlying asset's price implied by the VG process with $\varsigma = 0.1213$, $\nu = 0.1686$, $\vartheta = -0.1436$. The dashed line represents the probability density of the logarithm of terminal underlying asset's price implied by Merton's jump-diffusion model with parameters $\mu = 0.0391$, $\sigma = 0.1034$, $\lambda = 0.3283$, $\alpha = -0.1461$ and $\beta = 0.0384$, which are solved by the method provided in this section. Figure 3.2 illustrates the fact that the calibration method provided in this section can generate a proxy density which is closed to the target density.

Figure 3.2 demonstrate the error-reducing effect for our algorithm to price a vanilla call option whose underlying asset follows the VG process. We plot the real part of the integrand of the Fourier inversion, $e^{-ivk}\psi_T(v)$, as well as its forth derivative, $\frac{d^4}{dv^4}(e^{-ivk}\psi_T(v))$, because for one thing, we use the composite Simpson's rule, as suggested in Carr and Madan (1999), for quadrature. As a result, the numerical error committed is bounded by a constant multiple of the maximum absolute value of the fourth derivative of the integrand, as shown in Eq. (3). For another, the price of a call option should be a real number, and therefore we focus on the real part of the integrand.

The real part of the integrands of the Fourier inversion in the original Carr and Madan's method and our method with $H = 7$, $e^{-ivk}\psi_T(v)$ and $e^{-ivk}[\psi_T^{MJD,2}(v) + \psi_T^{RESIDUAL}(v)]$, are shown in panel (a) and (b), respectively. These two integrands are plotted in panel (c) with part of the magnitude larger than 20 being truncated. The real part of the fourth derivative of the integrands of the Fourier inversion in the original Carr and Madan's method and our method, $\frac{d^4}{dv^4}(e^{-ivk}\psi_T(v))$ and $\frac{d^4}{dv^4}\left(e^{-ivk}[\psi_T^{MJD,2}(v) + \psi_T^{RESIDUAL}(v)]\right)$, are shown in panel (d) and (e), respectively. These two integrands are plotted in panel (f) with part of the magnitude larger than 200,000 being truncated. All panels are based on the VG model with setting $S_0 = 100$, $K = 100$, $r = 0$, $T =$ 4 months, $\varsigma = 0.1213$, $\nu = 0.1686$, and $\vartheta = -0.1436$. The key point is that the amplitudes of the 4th derivative of the integrand for our method are significantly smaller than the amplitudes of the 4th derivative of the integrand for the original Carr and Madan's method. Since the numerical error committed by numerical integration with the Composite Simpson's rule is bounded above by the supremum of the 4th derivative, our method considerably improves the upper bound of quadrature error.

4 Numerical Results

We take the VG model, the Heston model, and Kou's jump-diffusion model for numerical examples. The parameter values used for the VG model are $S_0 = 100$, $K =$

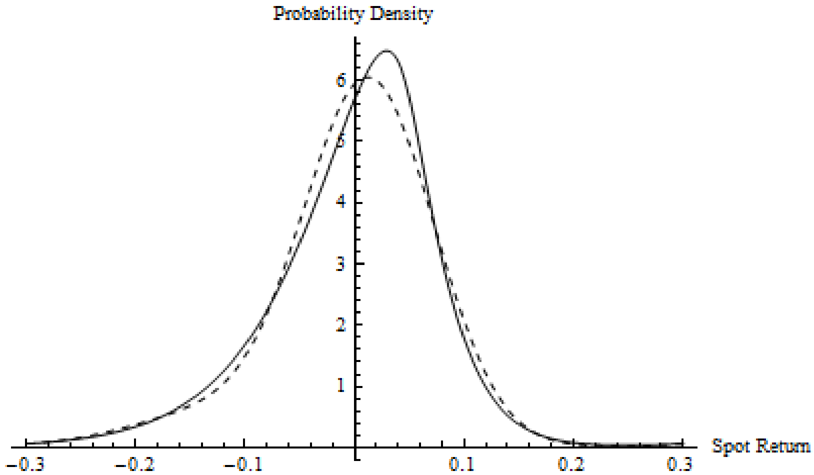

Figure 1: Comparison of the target and proxy densities.

The solid line represents the probability density implied by the VG process with parameters $\varsigma = 0.1213$, $\nu = 0.1686$, and $\vartheta = -0.1436$. The dashed line represents the probability density implied by Merton's jump-diffusion model with parameters $\mu = 0.0391$, $\sigma = 0.1034$, $\lambda = 0.3283$, $\alpha = -0.1461$ and $\beta = 0.0384$, which are solved by the method provided in this section. This figure illustrates the fact that the calibration method provided in this section can generate a proxy density which is closed to the target density.

100, $r = 0$, $T =$ 4 months, $\varsigma = 0.1213$, $\nu = 0.1686$, and $\vartheta = -0.1436$. The parameter values used for the Heston model are $S_0 = 100$, $K = 100$, $r = 0$, $T =$ 4 months, $\kappa = 1.49$, $\theta = 0.0671$, $\epsilon = 0.742$, $\rho = -0.571$, and $V_0 = 0.0262$. The parameter values used for Kou's jump-diffusion model are $S_0 = 100$, $K = 98$, $r = 0.05$, $T =$ 6 months, $\omega_1 = 10$, $\omega_2 = 5$, $\tilde{\lambda} = 1$, $p = 0.4$, $q = 0.6$, and $\tilde{\sigma} = 0.16$. The convergence rate for both models are shown in Figure 4.

Panel (b), (d), and (f) show that our algorithm does significantly improve the pricing accuracy since the number of grids needed to achieve a specific pricing accuracy for our method is much less than the original Carr and Madan's method. However, our method takes additional computational time since we need to find corresponding parameters for Merton's model as well as invoke H Black-Scholes formulas besides the original Carr and Madan's algorithm. Therefore, we have to demonstrate the two time-error plots, panel (a), (c), and (e), to analyze the convergence speed and computational time. Given the same computational time constraint, the pricing accuracy for our method is significantly better than the original Carr and Madan's method in both cases. Note that different H means different kinds of proxies are used. When a

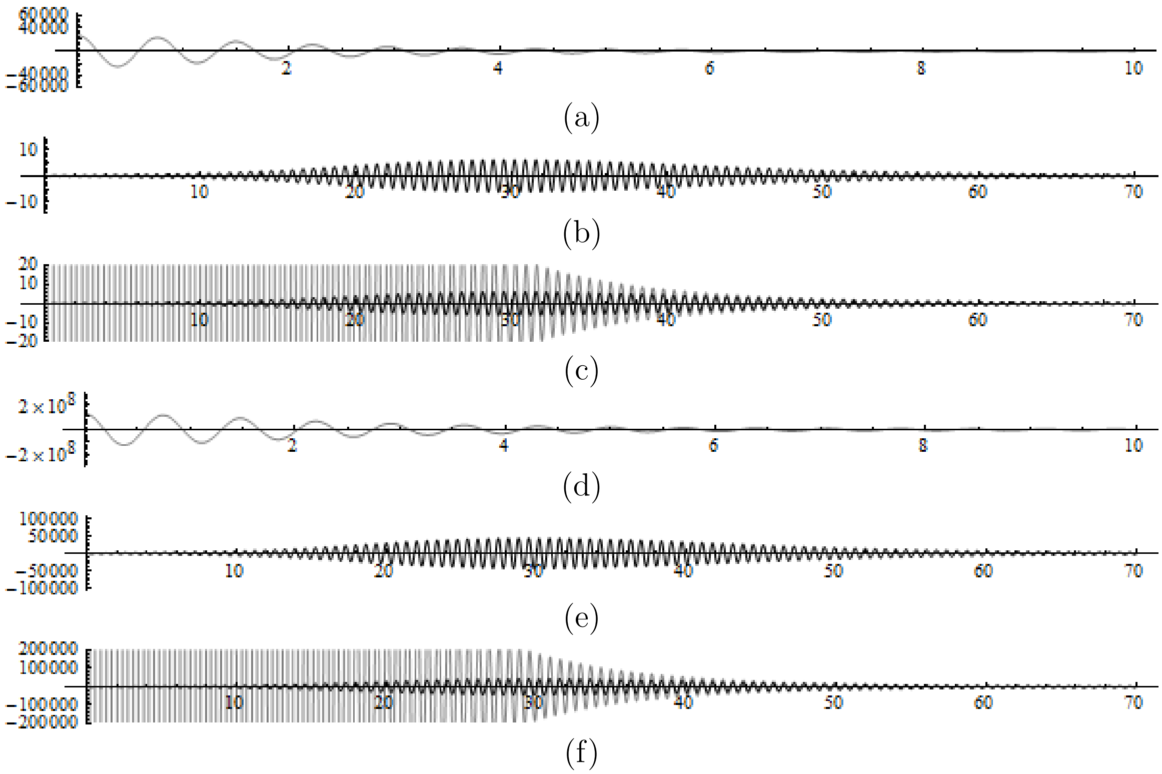

Figure 2: Comparison of the integrand in the Fourier inversion.

This figure demonstrates the error-reducing effect of our algorithm. The real part of the integrands of the Fourier inversion in the original Carr and Madan's method and our method with $H = 7$, $e^{-ivk}\psi_T(v)$ and $e^{-ivk}[\psi_T^{MJD,2}(v) + \psi_T^{RESIDUAL}(v)]$, are shown in panel (a) and (b), respectively. These two integrands are plotted in panel (c) with part of the magnitude larger than 20 being truncated. The real part of the fourth derivative of the integrands of the Fourier inversion in the original Carr and Madan's method and our method with $H = 7$, $\frac{d^4}{dv^4}(e^{-ivk}\psi_T(v))$ and $\frac{d^4}{dv^4}\left(e^{-ivk}[\psi_T^{MJD,2}(v) + \psi_T^{RESIDUAL}(v)]\right)$, are shown in panel (d) and (e), respectively. These two integrands are plotted in panel (f) with part of the magnitude larger than 200,000 being truncated. All panels are based on the VG model with setting $S_0 = 100$, $K = 100$, $r = 0$, $T =$ 4 months, $\varsigma = 0.1213$, $\nu = 0.1686$, and $\vartheta = -0.1436$. The key point is that the amplitudes of the 4th derivative of the integrand for our method are significantly smaller than the amplitudes of the 4th derivative of the integrand for the original Carr and Madan's method. Since the numerical error committed by numerical integration with the Composite Simpson's rule is bounded above by the supremum of the 4th derivative, our method considerably improves the upper bound of quadrature error.

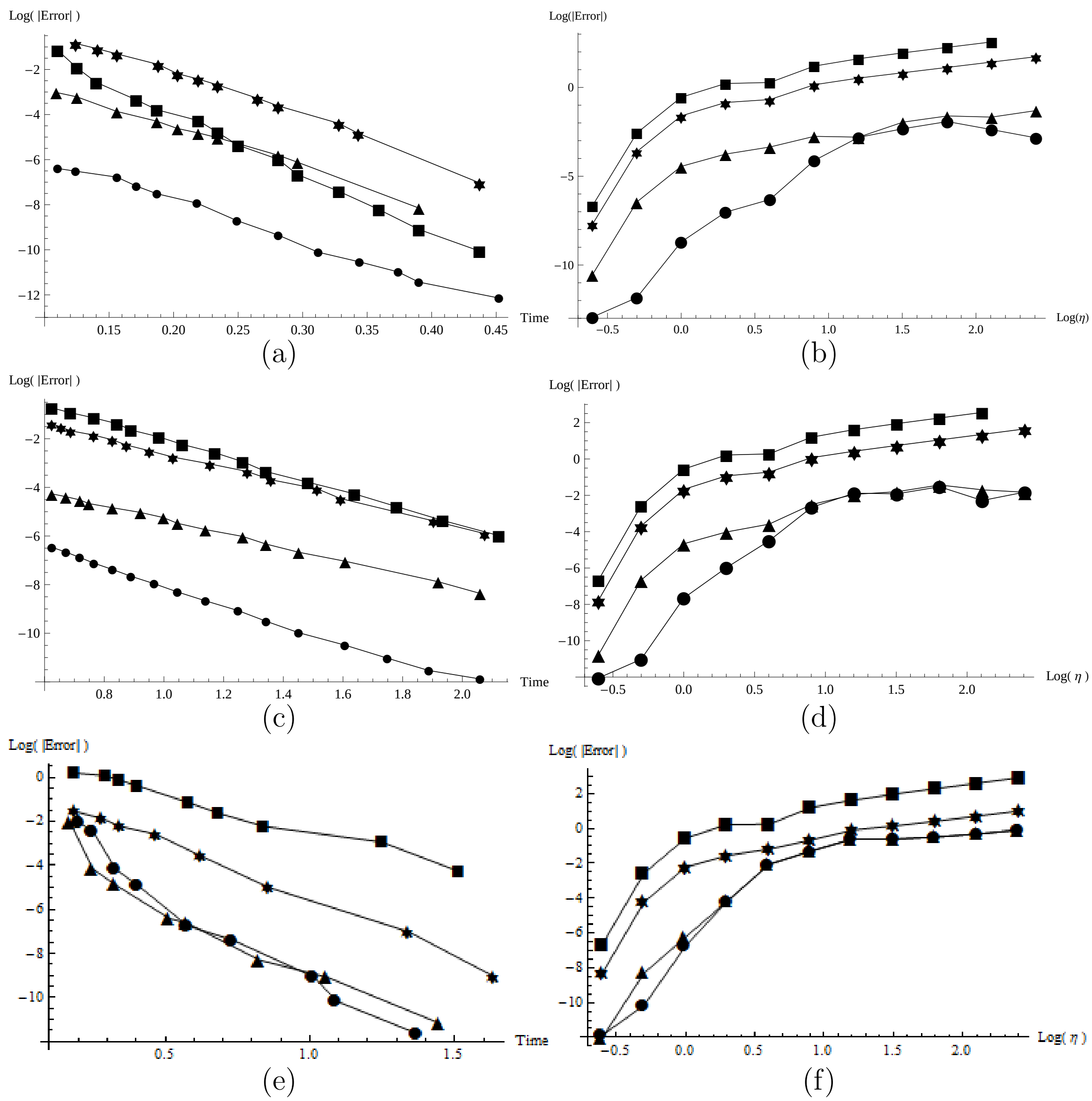

Figure 3: Convergence of pricing results.

The pricing errors are computed from, in panel (a) and (b), the VG model with $S_0 = 100$, $K = 100$, $r = 0$, $T =$ 4 months, $\varsigma = 0.1213$, $\nu = 0.1686$, $\vartheta = -0.1436$ and, in panel (c) and (d), the Heston model with $S_0 = 100$, $K = 100$, $r = 0$, $T =$ 4 months, $\kappa = 1.49$, $\theta = 0.0671$, $\epsilon = 0.742$, $\rho = -0.571$, $V_0 = 0.0262$ and, in panel (e) and (f), Kou's jump-diffusion model with $S_0 = 100$, $K = 98$, $r = 0.05$, $T =$ 6 months, $\omega_1 = 10$, $\omega_2 = 5$, $\tilde{\lambda} = 1$, $p = 0.4$, $q = 0.6$, $\tilde{\sigma} = 0.16$. In both panel (a), (c), and (e), lines are computational time plotted against the logarithm of the absolute pricing error. In both panel (b), (d), and (f), lines are $\log(\eta)$ plotted against the logarithm of the absolute pricing error. The squares denote the pricing results for the original Carr and Madan's method; the stars, triangles, and circles denote the pricing results for our method with $H = 1, 3$, and 7, respectively. The damping coefficient and the effective upper limit for integration are set to be 1.5 and 1024, respectively, as in Carr and Madan (1999). The pricing results implied by the original Carr and Madan's method with $N = 2^{20}$ quadrature points are used as the benchmarks.

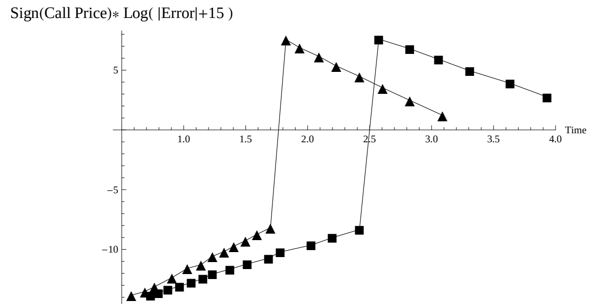

Figure 4: Comparison of negative pricing results with and without the proxy.

This figure demonstrates the negative pricing results for a deep out-of-the-money option. The x-axis denotes the computational time and the y-axis denotes product of sign function value of the call price and 15 plus the absolute pricing error. The squares and triangles denote the pricing results for the original Carr and Madan's method and our method with $H = 7$, respectively. We use "sign(Call Price) log(|Error| + 15)" as the unit of the y-axis because we want to make sure that the pricing result will be plotted above the x-axis when the corresponding call price is positive. The original Carr and Madan's method should use $\eta = 0.125$ in order to get positive pricing results; however, in our novel method, we can get positive pricing results with $\eta = 0.25$. The parameter values used for the Heston model are $S_0 = 100$, $K = 200$, $r = 0.03$, $T =$ 6 months, $\kappa = 2$, $\theta = 0.04$, $\epsilon = 0.5$, $\rho = -0.7$, and $V_0 = 0.04$. The damping coefficient and the effective upper limit for integration are set to be 1.5 and 1024, respectively. The pricing results implied by the original Carr and Madan's method with 2^{20} quadrature points are used as the benchmarks.

better proxy is chosen, the pricing accuracy will be better.

Furthermore, the original Carr and Madan's method often returns negative prices for deep out-of-the-money options (see e.g. Carr and Madan (2009)) because the integrand of the Fourier inversion for deep out-of-the-money options is highly oscillatory, and therefore deteriorates the convergence rate of quadrature. Our method can also alleviate this problem. We demonstrate this alleviation effect by the Heston model with $S_0 = 100$, $K = 200$, $r = 0.03$, $T =$ 6 months, $\kappa = 2$, $\theta = 0.04$, $\epsilon = 0.5$, $\rho = -0.7$, and $V_0 = 0.04$, which are the same as the parameters used in Carr and Madan (2009). Figure 4 shows that the pricing result for our method converges to positive prices faster than the original Carr and Madan's method. We use "sign(Call Price) log(|Error| + 15)" as the unit of the y-axis because we want to make sure that the pricing result will be plotted above the x-axis when the corresponding call price is positive Thus, we conclude that our method does alleviate the negative pricing result.

5 Conclusions

This paper provides a novel error-reducing method for the option valuation method studied by Carr and Madan (1999). Under the same computational time constraint, our method improves the pricing accuracy significantly. It is because the forth derivative of the integrand in the Fourier inversion has smaller amplitude for our method, and therefore the upper bound of numerical error is improved. Although we illustrate the benefits of our methodology by vanilla option only, our method can be extended to other kinds of financial derivatives with a suitable proxy. For example, the three kinds of payoffs suggested in Lee (2004). We also anticipate that the idea in our paper can be further extended to other kinds of numerical methods.

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